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Linear Algebra

An Introductory Approach

With 37 Illustrations

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To Timothy, Daniel, and Robert

Preface

Linear algebra is the branch of mathematics that has grown out of a theoretical study of the problem of solving systems of linear equations. The ideas that developed in this way have become part of the language of practically all of higher mathematics. They also provide a framework for applications in mathematical economics (linear programming), computer science, and the natural sciences.

This book is the fourth edition of a textbook designed for upper division courses in linear algebra. While it does not presuppose an earlier course, the pace of the discussion, and the emphasis on the theoretical development, make it best suited for students who have completed the calculus sequence.

For many students, this may be the first course in which proofs of the main results are presented on an equal footing with the methods for solving problems. The theoretical concepts are shown to emerge naturally from attempts to understand concrete problems. This connection is illustrated by worked examples in almost every section.

The subject of linear algebra is particularly satisfying in that the proofs of the main theorems usually contain the computational procedures needed to solve numerical problems. Many numerical exercises are included, which use all the essential ideas, and develop important techniques for problem-solving. There are also theoretical exercises, which provide opportunities for students to discover interesting things for themselves, to find variations on arguments used in the text, and to learn to write mathematical discussions in a clear and coherent way. Answers and hints for the theoretical problems are given in the back. Not all answers are given, however, to encourage students to develop their own procedures for checking their work.

A special feature of the book is the inclusion of sections devoted to applications of linear algebra. These are: §§8 and 9 on systems of equations, §10 on the geometrical interpretation of systems of linear equations, §§14 and 33 on finite symmetry groups in two and three dimensions, §34 on systems of first-order differential equations and the exponential of a matrix, and §35 on the composition of quadratic forms. They contain some historical remarks, along with self-contained expositions, based on the results in the text, of deeper results which illustrate the power and versatility of linear algebra. These sections are not required in the mainstream of the discussion and can either be put in a course or used for independent study.

Another feature of the book is that it provides an introduction to the axiomatic methods of modern algebra. The basic systems of algebra—groups, rings, and fields—all occur naturally in linear algebra. Their definitions and elementary properties are established as they come up in the discussion. A course based on this book can be followed by a thorough treatment of groups, rings, fields, and modules at the advanced undergraduate level. It should be noted that the basic properties of fields are developed in §2 and used in §3 to define vector spaces. Nothing is lost, however, by following a less abstract approach, omitting §2 and considering only vector spaces over the field F of real numbers in Chapters 2–5. It then becomes necessary to point out that the results in Chapters 2, 3 and 5 also hold (with the same proofs) for vector spaces over the field of complex numbers, before starting Chapter 7.

A course of one semester (or two quarters) should include Chapters 2–6, §§22–24 from Chapter 7, and §§30 and 31 from Chapter 9, with some introductory remarks at the beginning about fields and mathematical induction from §2.

A year course will also contain the theory of the rational and Jordan canonical forms, beginning with §25. The proof of the elementary divisor theorem (§29) is based on the theory of dual vector spaces. The algorithm for determining the invariant factors of a matrix with polynomial entries is not included, however; this belongs in the theory of modules over principal ideal domains, and is left for a later course. Other topics to be included in a year course based on the book are tensor products of vector spaces, unitary transformations and the spectral theorem, and the sections on applications mentioned earlier.

It is a pleasure to acknowledge the encouragement I have received for this project from students in my classes at Wisconsin and Oregon. Their comments, and thoughtful reviews from instructors who taught from previous editions, have led to improvements from one edition to the next, including corrections, revisions, and additional exercises in this new edition. I am also grateful for the interest and support of my family.

Eugene, Oregon
February 12, 1984

Charles W. Curtis

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Chapter 1

Introduction to Linear Algebra

Mathematical theories are not invented spontaneously. The theories that have proved to be useful have their beginnings, in most cases, in special problems, which are difficult to understand or to solve without a grasp of the underlying principles. This chapter begins with two such problems, which have a common set of underlying principles. Linear algebra, which is the study of vector spaces, linear transformations, and matrices, is the result of trying to understand the common features of these and other similar problems.

1. SOME PROBLEMS WHICH LEAD TO LINEAR ALGEBRA

This section should be read quickly the first time through, with the objective of getting some motivation, but not a thorough understanding of the details.

Problem A. Everyone is familiar from elementary algebra with the problem of solving two linear equations in two unknowns, for example,

(1.1)
$$\begin{aligned} 3x - 2y &= 1 \\ x + y &= 2. \end{aligned}$$

We solve such a system by eliminating one unknown, for example, by multiplying the second equation by 2 and adding it to the first. This yields $5x = 5$, and $x = 1$. Substituting back in the first equation, we see that

$$3 - 2y = 1,$$

or $y = 1$. The solution of the original system of equations is the pair of numbers (1, 1), and there is only one solution.

If we look at the original problem from a geometrical point of view, then it becomes clear what is happening. The equations

$$3x - 2y = 1 \qquad \text{and} \qquad x + y = 2$$

are equations of straight lines in the (x, y) plane (see Figure 1.1). The

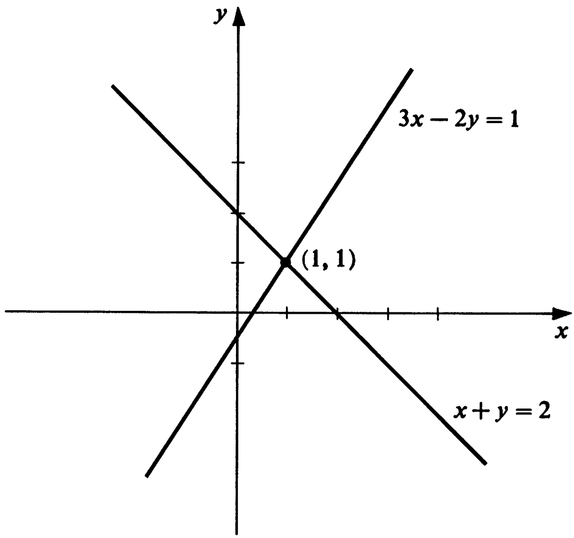

FIGURE 1.1

solution we obtained to the system of equations expresses the fact that the two lines intersect in a unique point, which in this case is (1, 1).

Now let us consider a similar problem, involving two equations in three unknowns.

$$\begin{aligned} x + 2y - z &= 1 \\ 2x + y + z &= 0. \end{aligned} \tag{1.2}$$

This time it isn't so easy to eliminate the unknowns. Yet we see that if z is given a particular value, say 0, we can solve the resulting system

$$\begin{aligned} x + 2y &= 1 \\ 2x + y &= 0 \end{aligned} \qquad (z = 0)$$

and obtain $x = -\frac{1}{3}$, $y = \frac{2}{3}$; so a solution of the original system (1.2) is $(-\frac{1}{3}, \frac{2}{3}, 0)$. But this time there are many other solutions; for example, letting $z = 1$, we obtain a solution $(-\frac{4}{3}, \frac{5}{3}, 1)$. In fact we can obtain a solution for every value of z. The puzzle of why we get so many solutions is answered again by a geometrical interpretation. The equations in (1.2) are equations of planes in (x, y, z) coordinates (see Figure 1.2).

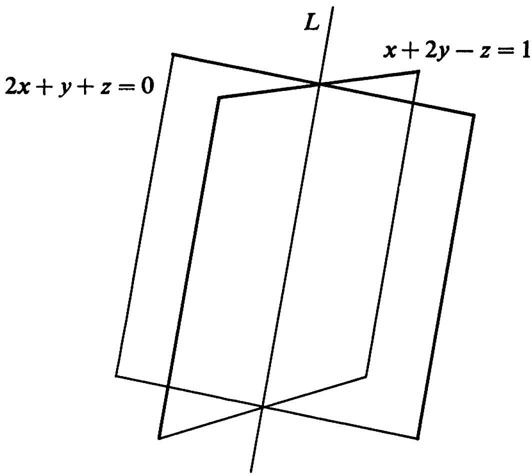

FIGURE 1.2

This time the solutions, viewed as points in (x, y, z) coordinates, correspond to the points on the line of intersection L of the two planes.

The second problem already shows that more work is needed to describe the solutions of the system (1.2) and to give a method for finding all of them. But why stop there? In many types of problems, a solution is needed to systems of equations in more than two or three unknowns; for example,

$$\begin{aligned} x + 2y - 3z + t &= 1 \\ x + y + z + t &= 0. \end{aligned} \tag{1.3}$$

In this case and in other more complicated ones, geometric intuition is not available (for most of us, anyway) to give a picture of what the solution should be. At the same time such examples are not farfetched from the applied point of view since, for example, a formula describing the temperature t at a point (x, y, z) involves four variables. One of the tasks of linear algebra is to provide a framework for discussing problems of this nature.

Problem B. In calculus, a familiar type of question is to find a function $y = f(x)$ if its derivative $y' = Df(x)$ is known. For example, if y' is the constant function 1, we know that $y = x + C$, where C is a constant. This type of problem is called a differential equation. Problems in both mechanics and electric circuits lead to differential equations such as

$$y'' + m^2 y = 0, \tag{1.4}$$

where m is a positive constant, and y'' is the second derivative of the unknown function y. This time the solution is not so obvious.

Checking through the rules for differentiation of the elementary functions, the functions $y = A \sin mx$ or $y = B \cos mx$ are recognized as solutions, where A and B are arbitrary constants. But here, as in the case of the equations (1.2) or (1.3), there are many other solutions, such as $C \sin (mx + D)$, where C and D are constants. The problem in this case is to find a clear description of all possible solutions of differential equations such as (1.4).

Now let us see what if anything the two types of problems have in common, beyond the fact that both involve solutions of equations. In the case of the equations (1.3), for example, we are interested in ordered sets of four numbers $(\alpha, \beta, \gamma, \delta)$† which satisfy the equations (1.3) when substituted for x, y, z, and t, respectively. We shall call $(\alpha, \beta, \gamma, \delta)$ a *vector* with four entries; later (in Chapter 2) we shall define vectors $(\alpha_1, \ldots, \alpha_n)$ with n entries $\alpha_1, \ldots, \alpha_n$ for $n = 1, 2, 3, \ldots$. A vector $(\alpha, \beta, \gamma, \delta)$ which satisfies the equations (1.3) will be called a *solution vector* of the system (1.3) [or simply a *solution* of (1.3)]. The following statements about solutions of (1.3) can now be made.

(*i*) If $u = (\alpha, \beta, \gamma, \delta)$ and $u' = (\alpha', \beta', \gamma', \delta')$ are both solutions of (1.3), then

$$(\alpha - \alpha', \ \beta - \beta', \ \gamma - \gamma', \ \delta - \delta')$$

is a solution of the system of *homogeneous equations* (obtained by setting the right-hand side equal to zero),

$$\begin{aligned} x + 2y - 3z + t &= 0 \\ x + y + z + t &= 0. \end{aligned} \tag{1.5}$$

(*ii*) If $u = (\alpha, \beta, \gamma, \delta)$ and $u' = (\alpha', \beta', \gamma', \delta')$ are solutions of the homogeneous system (1.5), then so are

$$(\alpha + \alpha', \beta + \beta', \gamma + \gamma', \delta + \delta')$$

and

$$(\lambda\alpha, \ \lambda\beta, \ \lambda\gamma, \ \lambda\delta)$$

for an arbitrary number λ.

† See the list of Greek letters on p. 332.

(*iii*) Suppose $u_0 = (\alpha, \beta, \gamma, \delta)$ is a fixed solution of the nonhomogeneous system (1.3). Then an arbitrary solution v of the nonhomogeneous system has the form

(1.6) $$v = (\alpha + \xi, \beta + \eta, \gamma + \lambda, \delta + \mu)$$

where $(\xi, \eta, \lambda, \mu)$ is an arbitrary solution of the homogeneous system (1.5).

Statements (i) and (ii) are verified by direct substitution in the equations. To check statement (iii), suppose first that $v = (a, b, c, d)$ is an arbitrary solution of the system (1.3). By statement (i), recalling that $u_0 = (\alpha, \beta, \gamma, \delta)$ is our fixed solution of (1.3), we see that

$$(a - \alpha, b - \beta, c - \gamma, d - \delta)$$

is a solution of the homogeneous system, and setting $\xi = a - \alpha$, $\eta = b - \beta$, $\lambda = c - \gamma$, $\mu = d - \delta$, we have

$$v = (a, b, c, d) = (\alpha + \xi, \beta + \eta, \gamma + \lambda, \delta + \mu)$$

as required. A further argument (which we omit) shows that every vector of the form (1.6) is actually a solution of the nonhomogeneous system (1.3).

What have we learned from all this? First, that facts about the solutions of the equations can be expressed in terms of certain operations on vectors:

$$(\alpha, \beta, \gamma, \delta) + (\alpha', \beta', \gamma', \delta') = (\alpha + \alpha', \beta + \beta', \gamma + \gamma', \delta + \delta')$$

and $$\lambda(\alpha, \beta, \gamma, \delta) = (\lambda\alpha, \lambda\beta, \lambda\gamma, \lambda\delta).$$

Thus we may add two vectors and obtain a new one, and multiply a vector by a number, to give a new vector. In terms of these operations we can then describe the problem of solving the system (1.3) as follows: (1) We find one solution u_0 of the nonhomogeneous system; (2) an arbitrary solution (often called the *general solution*) v of the nonhomogeneous system is given by

$$v = u_0 + u,$$

where u ranges over the set of solutions of the homogeneous system; (3) we find all solutions of the homogeneous system (1.5).

Now let us turn to the differential equation (1.4). We see first that if the functions f_1 and f_2 are solutions of the equation (that is, $f_1'' + m^2 f_1 = 0$, $f_2'' + m^2 f_2 = 0$), then $f_1 + f_2$ and λf_1 are also solutions, where λ is an arbitrary number. But this is the same situation we encountered in statement (ii) above, whereas now we have functions in place of vectors. Let's go a little further. Suppose we take a nonhomogeneous differential equation (by analogy with the linear equations),

(1.7) $$y'' + m^2 y = F,$$

where F is some fixed function. Then we see that all three of our statements about Problem A are satisfied. The difference of two solutions of the nonhomogeneous system (1.7) is a solution of the homogeneous system. The solutions of the homogeneous system satisfy statement (ii), if the operations of adding functions and multiplying functions by constants replace the corresponding operations on vectors. Finally, an arbitrary solution of the nonhomogeneous system is obtained from a particular one f_0 by adding to f_0 a solution of the homogeneous system.

It is reasonable to believe that the analogous behavior of the solutions of Problems A and B is not a pure coincidence. The common framework underlying both problems will be provided by the concept of a vector space, to be introduced in Chapter 2. Before beginning the study of vector spaces, we have one more introductory section in which we review facts about sets and numbers which will be needed.

2. NUMBER SYSTEMS AND MATHEMATICAL INDUCTION

We assume familiarity with the real number system, as discussed in earlier courses in college algebra, trigonometry, or calculus. It will turn out, however, that number systems other than the real numbers (such as complex numbers) also play an important role in linear algebra. At the same time, not all the detailed facts about these number systems (such as absolute value and polar representation of complex numbers) are needed until later in the book.

In order to have a clearly defined starting place, we shall give a set of axioms for a number system, called a *field*, which will provide a firm basis for the theory of vector spaces to be discussed in the next chapter. The real number system is an example of a field. We first have to recall a few of the basic ideas about sets.

We use the word *set* as synonymous with "collection" or "family" of objects of some sort.

Let X be a set of objects. For a given object x, either x belongs to the set X or it does not. If x belongs to X, we write $x \in X$ (read "x is an element of X" or "x is a member of X"); if x does not belong to X, we write $x \notin X$.

A set Y is called a *subset* of a set X if, for all objects, y, $y \in Y$ implies $y \in X$. In other words, every element of Y is also an element of X. If Y is a subset of X, we write $Y \subset X$. If $Y \subset X$ and $X \subset Y$, then we say that the sets X and Y are equal and write $X = Y$. Thus two sets are equal if they contain exactly the same members.

It is convenient to introduce the set containing no elements at all.

We call it the empty set, and denote it by $\varnothing$. Thus, for every object x, $x \notin \varnothing$. For example, the set of all real numbers x for which the inequalities $x < 0$ and $x > 1$ hold simultaneously is the empty set. The reader will check that from our definition of subset it follows logically that the empty set $\varnothing$ is a subset of every set (why?).

There are two important constructions which can be applied to subsets of a set and yield new subsets. Suppose U and V are subsets of a given set X. We define $U \cap V$ to be the set consisting of all elements belonging to both U and V and call $U \cap V$ the *intersection* of U and V. *Question:* What is the intersection of the set of real numbers x such that $x > 0$ with the set of real numbers y such that $y < 5$? If we have many subsets of X, their intersection is defined as the set of all elements which belong to all the given subsets.

The second construction is the *union* $U \cup V$ of U and V; this is the subset of X consisting of all elements which belong either to U or to V. (When we say "either ... or," it is understood that we mean "either ... or ... or both.")

It is frequently useful to illustrate statements about sets by drawing diagrams. Although they have no mathematical significance, they do give us confidence that we are making sense, and sometimes they suggest important steps in an argument. For example, the statement $X \subset Y$ is illustrated by Figure 1.3. In Figure 1.4 the shaded portion indicates $U \cup V$, while the cross-hatched portion denotes $U \cap V$.

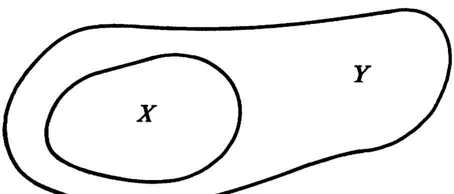

FIGURE 1.3

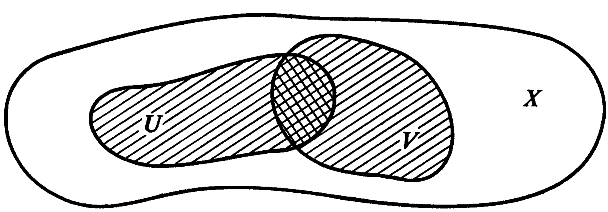

FIGURE 1.4

Examples and Some Notation

We shall often use the notation

$$\{x \in X \mid x \text{ has the property } P\}$$

to denote the set of objects x in a set X which have some given property P. Thus, if R denotes the set of real numbers, $\{x \in R \mid x < 5\}$ denotes the set of all real numbers x such that $x < 5$. Using this notation, the set of solutions of the system of equations (1.1) is described by the statement

$$\{(x, y) \mid 3x - 2y = 1\} \cap \{(x, y) \mid x + y = 2\} = \{(1, 1)\},$$

where $\{(1, 1)\}$ denotes the set containing the single point $(1, 1)$. In general, we shall use the notations $\{a\}$, $\{a, b\}$, $\{a_i\}$, etc., to denote the sets whose members are a, a and b, and a_i, $i = 1, 2, \ldots$, respectively. Examples of sets are plentiful in geometry. Lines and planes are sets of points; the intersection of two lines in a plane is either the empty set (if the lines are parallel) or a single point; the intersection of two planes is either $\varnothing$ or a line.

We now introduce the kind of number system which will be the starting point of our development of vector spaces in Chapter 2.

(2.1) DEFINITION. A *field* is a mathematical system F consisting of a nonempty set F together with two operations, addition and multiplication, which assign to each pair of elements† $\alpha, \beta \in F$ uniquely determined elements $\alpha + \beta$ and $\alpha\beta$ (or $\alpha \cdot \beta$) of F, such that the following conditions are satisfied, for $\alpha, \beta, \gamma \in F$.

1. $\alpha + \beta = \beta + \alpha$, $\alpha\beta = \beta\alpha$ (*commutative laws*).
2. $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$, $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ (*associative laws*).
3. $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$ (*distributive law*).
4. There exists an element 0 in F such that $\alpha + 0 = \alpha$ for all $\alpha \in F$.
5. For each element $\alpha \in F$ there exists an element $-\alpha$ in F such that $\alpha + (-\alpha) = 0$.
6. There exists an element $1 \in F$ such that $1 \neq 0$ and such that $\alpha \cdot 1 = \alpha$ for all $\alpha \in F$.
7. For each nonzero $\alpha \in F$ there exists an element $\alpha^{-1} \in F$ such that $\alpha\alpha^{-1} = 1$.

The first example of a field is the system of real numbers R, with the usual operations of addition and multiplication.

The set of integers $Z = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is a subset of R, which is *closed* under the operations of R, in the sense that if m and n belong to Z, then their sum $m + n$ and product mn (which are defined in

† See the list of Greek letters on page 332.

terms of the operations of the larger set R) again belong to Z. But Z is not a field with respect to these operations, since the axiom [2.1(7)] fails to hold. For example, $2 \in Z$ and $2^{-1} \in R$, but $2^{-1} \notin Z$, and in fact there is no element $m \in Z$ such that $2m = 1$. This example suggests the following definition.

(2.2) DEFINITION. A *subfield* F_0 of a field F is a subset F_0 of F, which is closed under the operations defined on F, and which satisfies all the axioms [2.1(1)–(7)] relative to these operations. (Here *closed* means that if $\alpha, \beta \in F_0$, then $\alpha + \beta \in F_0$ and $\alpha\beta \in F_0$.)

Although the integers Z are not a subfield of R, the set of rational numbers Q is a subfield. We recall that the set of rational numbers Q consists of all real numbers of the form

$$mn^{-1}$$

where $m, n \in Z$ and $n \neq 0$. It takes some work to show that the rational numbers actually are a subfield of R. The kinds of things that have to be checked are discussed later in this section.

The fields of both real numbers and rational numbers are *ordered fields* in the sense that each one contains a subset P (called the set of *positive elements*) with the properties

1. $\alpha, \beta \in P$ implies $\alpha + \beta$ and $\alpha\beta \in P$.
2. For each α in the field, one and only one of the following possibilities holds:

$$\alpha \in P, \qquad \alpha = 0, \qquad -\alpha \in P.$$

The field of complex numbers C (to be discussed more fully in Chapter 6) is not an ordered field. It is defined as the set of all pairs of real numbers (α, β), $\alpha, \beta \in R$, where two pairs are defined to be equal, $(\alpha, \beta) = (\alpha', \beta')$, if and only if $\alpha = \alpha'$ and $\beta = \beta'$. The operations in C are defined as follows:

$$(\alpha, \beta) + (\alpha', \beta') = (\alpha + \alpha', \beta + \beta')$$

and

$$(\alpha, \beta)(\alpha', \beta') = (\alpha\alpha' - \beta\beta', \alpha\beta' + \beta\alpha').$$

The proof that the axioms for a field are satisfied is given in Chapter 6.

Consider the set of all complex numbers of the form $\{(\alpha, 0), \alpha \in R\}$. These have the properties that for all $\alpha, \beta \in R$,

(2.3)
$$(\alpha, 0) + (\beta, 0) = (\alpha + \beta, 0),$$
$$(\alpha, 0)(\beta, 0) = (\alpha\beta, 0).$$

It can be checked that these form a subfield R' of C. Moreover, the rule or correspondence $\alpha \leftrightarrow (\alpha, 0)$ which assigns to each $\alpha \in R$ the complex number $(\alpha, 0) \in R'$ has the properties that (a) $(\alpha, 0) = (\alpha', 0)$ if and only if $\alpha = \alpha'$ and that (b) it preserves the operations in the respective number systems, according to the equations (2.3) above. A correspondence between two fields with the properties (a) and (b) is called an *isomorphism*, the word coming from Greek words meaning "to have the same form." The existence of the isomorphism $\alpha \leftrightarrow (\alpha, 0)$ between the fields R and R' means that if we ignore all other properties not given in the definition of a field, then R and R' are indistinguishable. Thus we shall identify the elements of R with the corresponding elements of R', and in this sense we have the following inclusions among the number systems defined so far:

$$Z \subset Q \subset R \subset C.$$

We come now to the principle of mathematical induction, which we shall take as an axiom about the set of positive integers $Z^+ = \{1, 2, \ldots\}$. It is impossible to overemphasize its importance for linear algebra. Almost all the deeper results in this book depend on it in one way or another.

(2.4) PRINCIPLE OF MATHEMATICAL INDUCTION. *Suppose that for each positive integer n there corresponds a statement $E(n)$ which is either true or false. Suppose, further, (A) $E(1)$ is true; and (B), if $E(n)$ is true then $E(n + 1)$ is true, for all $n \in Z^+$. Then $E(n)$ is true for all positive integers n.*

Some equivalent forms of the principle of induction are often useful, and we shall list some of them explicitly.

(2.5)A *Let $\{E(n)\}$ be a family of statements defined for each positive integer n. Suppose* (a) *$E(1)$ is true, and* (b) *if $E(r)$ is true for all positive integers $r < n$, then $E(n)$ is true. Then $E(n)$ is true for all positive integers n.*

In discussions where mathematical induction is involved, the statement which plays the role of $E(n)$ will often be called the *induction hypothesis.*

An equivalent form of (2.4) will be used later, especially in Chapter 6. For a discussion of how this statement can be shown to be equivalent to (2.4), see the book of Courant and Robbins listed in the bibliography.

(2.5)B WELL-ORDERING PRINCIPLE. *Let M be a nonempty set of positive integers. Then M contains a least element, that is, an element m_0 in M exists, which satisfies the condition $m_0 \leq m$, for all $m \in M$.*

Examples of Mathematical Induction

Example A. We shall prove the formula

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}, \tag{2.6}$$

for all positive integers n. The induction hypothesis is the statement (2.6) itself. For $n = 1$, the statement reads $1 = 1(2)/2$. Now suppose the statement (2.6) holds for some n. We have to show it is true for $n + 1$. Substituting the right-hand side of (2.6) for $1 + 2 + \cdots + n$, we have

$$\begin{aligned} 1 + 2 + \cdots + n + (n+1) &= \frac{n(n+1)}{2} + (n+1) \\ &= (n+1)\left(\frac{n}{2} + 1\right) = \frac{(n+1)(n+2)}{2}, \end{aligned}$$

which is the statement (2.6) for $n + 1$. By the principle of mathematical induction, we conclude that (2.6) is true for all positive integers n.

Example B. For an arbitrary real number $x \neq 1$,

$$1 + x + x^2 + \cdots + x^{n-1} = \frac{1 - x^n}{1 - x}. \tag{2.7}$$

Again we let (2.7) be the induction hypothesis $E(n)$. For $n = 1$, the statement is

$$1 = \frac{1 - x}{1 - x}.$$

Assume (2.7) for some n. Then using (2.7), we have

$$\begin{aligned} 1 + x + x^2 + \cdots + x^n &= (1 + x + \cdots + x^{n-1}) + x^n \\ &= \frac{1 - x^n}{1 - x} + x^n \\ &= \frac{1 - x^n + x^n(1 - x)}{1 - x} \\ &= \frac{1 - x^{n+1}}{1 - x}, \end{aligned}$$

and the statement (2.7) holds for all n.

We conclude this chapter with some properties of an arbitrary field F. These are all familiar facts about the system of real numbers, but the reader may wish to study some of them closely, to check that they depend only on the axioms for a field. Besides the axioms, the *principle of substitution* is frequently used. This simply means that in a field F, we