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Invariant Theory

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FOREWORD

This symposium is an outgrowth of a conference on Combinatorics and Invariant Theory held at West Chester University in the Summer of 1985. We felt at that time that a collection of well placed expository papers leading directly to the heart of current research in Invariant Theory would serve an useful purpose. We hope that the present volume has, in some measure, achieved that aim.

We would like to thank Dr. Frank Grosshans for his informative INTRODUCTION which unifies the individual papers into an organized whole.

Editor

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INTRODUCTION

Invariant theory was developed in the nineteenth century by Boole, Cayley, Clebsch, Gordan, Hilbert, Sylvester, and others. It has been studied intermittently ever since. In recent times, newly developed techniques from algebraic geometry and combinatorics have been applied with great success to some of its outstanding problems. This has moved invariant theory, once again, to the forefront of mathematical research.

In this introduction, we shall introduce the main problems of invariant theory and show how the papers in this collection are related to them. We begin with the necessary definitions.

Let K be an infinite field. Let V be a finite-dimensional vector space over K and let $\{e_1,...,e_n\}$ be any basis for V. Let $K[x_1,...,x_n]$ be the polynomial algebra in n variables over K. A function $f: V \longrightarrow K$ is called a <u>polynomial function</u> on V if there is a polynomial p in $K[x_1,...,x_n]$ such that for every v in V, $v = \sum_{i=1}^{n} a_i e_i$, we have $f(v) = p(a_1,...,a_n)$. We shall denote the algebra of all polynomial functions on V by K[V].

Next, let G be a group and suppose that G acts on V via some representation. We say that a polynomial function f on V is <u>invariant</u> with respect to G if $f(g\cdot v) = f(v)$ for all g in G and v in V. The set of all such invariant polynomials forms an algebra which we shall denote by $K[V]^G$.

Invariant theory covers a wide range of highly specialized problems and techniques. One way to impose some unity on this study is to use the notion of an orbit. Let v be any point in V; the <u>orbit</u> of v with respect to the action of G consists of all the points $g\cdot v$ where g is any element of G. We shall denote this orbit by $G\cdot v$. By definition, a polynomial f in $k[V]^G$ is constant on any orbit. The orbit $G\cdot v$ is called <u>separated</u> if $G\cdot v = \{v' \in V: f(v') = f(v) \text{ for all } f \in K[V]^G\}$. In general, not all orbits are separated but the separated orbits often form a large subset of V, e.g., one that is

Zariski-open and dense in V. We may take the beginning point of invariant theory to be the study of these separated orbits. There are three closely connected questions which we may state (somewhat vaguely) as follows:

(A) What is the structure of the algebra $K[V]^G$? This question starts with finite generation: are there elements $f_1, \ldots, f_m$ in $K[V]^G$ so that each element in $K[V]^G$ is a polynomial in $f_1, \ldots, f_m$? If so, the question as to whether an orbit $G \cdot v$ is separated looks easier, at least, since $\{v' \in V: f(v') = f(v)$ for all $f \in K[V]^G\} = \{v' \in V: f_i(v') = f_i(v)$ for $i = 1, \ldots, m\}$.

(B) How may the elements in $K[V]^G$ be used to define canonical forms on V? This question may be taken as a type of the same question encountered in matrix algebra or as one involving structure on the class of orbits. As an example of the latter, we might ask whether the separated orbits constitute an algebraic variety?

(C) How can invariant polynomial be constructed? In a sense, this question goes beyond the existence questions raised in Problems (A) and (B) and asks whether the generators of $K[V]^G$ can be written down explicitly or if canonical forms can be defined by certain explicitly determined invariant polynomials.

Example 1. Let $M_n(K)$ be the vector space consisting of all nxn matrices over K. The group $GL_n(K)$ acts on $M_n(K)$ via $g \cdot X = gXg^{-1}$. Given any matrix X in $M_n(K)$, we may determine its characteristic polynomial. The coefficients of the characteristic polynomial may be considered to be polynomial functions on $M_n(K)$. As such, they are invariant with respect to the action of $GL_n(K)$ and, indeed, may be shown to generate the algebra of invariant polynomials. Furthermore, these coefficients are algebraically independent, so the algebra of invariant polynomials is a polynomial algebra in n variables. The separated orbits are precisely the orbits of those X in $M_n(K)$ which have n distinct eigen-values.

Example 2. Let us fix a non-negative integer d and let V_d be the vector space over $\mathbb{C}$ consisting of all binary forms of degree d in the variables x and y. The space V_d has a basis $\binom{d}{i} x^{d-i} y^i$ for i = 0,1,...,d. The group $G = SL_2(\mathbb{C})$ acts (via multiplication on the left) on the vector space $\mathbb{C}^2$ consisting of all 2x1 column matrices. This gives an action of G on $\mathbb{C}[x,y]$. In particular, we have

$$(a_{ij})\cdot x = a_{22}x - a_{12}y \text{ and } (a_{ij})\cdot y = -a_{21}x + a_{11}y.$$

Relative to this action, we have an action of G on V_d and, so, an algebra of invariant polynomials $\mathbb{C}[V_d]^G$. Hilbert showed that the separated orbits in V_d are those of forms f in V_d which have no factor of multiplicity $\geq d/2$. He achieved this result without ever writing down the generators of $\mathbb{C}[V_d]^G$; this theme was taken up in recent times by Mumford and has developed into modern geometric invariant theory. The study of canonical forms in $\mathbb{C}[V_d]^G$ is somewhat different and we postpone our discussion of it.

The group G also acts on the product $V_d \times \mathbb{C}^2$ via $g\cdot(f,v) = (g\cdot f, g\cdot v)$. The invariants of this action were called covariants in the nineteenth century. Covariants were of interest in themselves and also for their applications to the study of $\mathbb{C}[V_d]^G$ and canonical forms in V_d . For example, Gordon proved in 1868 that $\mathbb{C}[V_d]^G$ is a finitely generated algebra over $\mathbb{C}$ by using a constructive argument involving covariants called transvectants.

Question (A): structure of $K[V]^G$

We mentioned that Gordan proved that $\mathbb{C}[V_d]^G$ is finitely generated. For arbitrary groups G and rational representations V, much more is known now, mainly due to the work of Hilbert, Weyl, Schiffer, Mumford, Nagata, and Haboush. Indeed, if G is a semi-simple algebraic group, then $K[V]^G$ is always finitely generated. Furthermore, if K has characteristic 0, then Hochster and

Roberts proved that $K[V]^G$ is Cohen-Macaulay, i.e., a free module over a polynomial subalgebra. (More is known; $K[V]^G$ is actually a Gorenstein ring but not, in general, a complete intersection.)

If G is not semi-simple (or reductive), then $K[V]^G$ may or may not be finitely generated. K. Pommerening ("Invariants of unipotent groups") summarizes the current state of our knowledge, here. Interestingly enough, one critical idea in this question comes from an interpretation of covariants in the study of binary forms. (In this setting, it is due to M. Roberts and dates to 1861.) Let $U = \{(a_{ij}) \in SL_2(\mathbf{C}) : a_{21} = 0,\ a_{11} = a_{22} = 1\}$. The algebras $\mathbf{C}[V_d]^U$ and $\mathbf{C}[V_d \times \mathbf{C}^2]^G$ are isomorphic. Thus, polynomials fixed by U (called semi-invariants) were used to construct covariants and, conversely, the "leading term" of any covariant gave a semi-invariant. With this isomorphism at hand, the finite generation (and Cohen-Macaulay property) of the algebra $\mathbf{C}[V_d]^U$ now follow from the analogous facts for $SL_2(\mathbf{C})$.

In Example 1, above, we considered the action of $GL_n(K)$ on $M_n(K)$ given by $g \cdot X = gXg^{-1}$. Let $D_n(K)$ consist of all the diagonal matrices in $M_n(K)$. The permutation group on n letters acts on $D_n(K)$ in the natural way. Furthermore, the invariants of $GL_n(K)$ on $M_n(K)$ are isomorphic to the invariants of the permutation group acting on $D_n(K)$. E. Formanek ("The invariants of nxn matrices") considers the action of $GL_n(K)$ on $M_n(K) \times M_n(K) \times \ldots \times M_n(K)$ given by $g \cdot (X_1, X_2, \ldots, X_r) = (gX_1 g^{-1}, gX_2 g^{-1}, \ldots, gX_r g^{-1})$. The generators for the algebra of invariant polynomials may be given explicitly using the trace function, but now there are relations among the generators. These relations give information about nilalgebras and, more generally, algebras satisfying polynomial identities. As would be expected, permutation groups play an important role in this study.

Question (B): canonical forms

We mentioned earlier that for the action of $G = SL_2(\mathbf{C})$ on V_d, the separated orbits in V_d are those of forms having no linear factor of multiplicity $\geq d/2$. For such forms, a good structure (in the algebraic-geometric sense) can be placed on the class of orbits. The canonical form problem for binary forms is concerned with writing an $f \in V_d$ as $\sum_{i=1}^{m} (a_i x+b_i y)^d$. This was studied with great success by Sylvester but there are still some unresolved questions. A. Lascoux ("Forme canonique d'une forme binaire") discusses the case where d is odd and J.P.S. Kung ("Canonical forms for binary forms of even degree") the case where d is even. The information coming from the invariant polynomials in $\mathbb{C}[V_d]^G$ is not sufficient to address the questions surrounding these canonical forms. Rather, the key to unlocking the questions comes from the use of certain explicitly given covariants, such as the catalecticant, the canonizant, and certain covariants discovered by Gundelfinger.

Much the same situation arises for the action of SL(V) on the spaces $\bigwedge^k(V)$. Let us illustrate this in the case where dim V = 6 and $SL_6(C)$ acts on $\bigwedge^3(V)$. In this case, Schouten discovered four canonical forms, namely: (I) bcd, (II) bcd + bef, (III) bcd + bfg + cef, (IV) bcd + efg. (Here, the symbols b,c,d,e,f,g denote linearly independent vectors in V and the "wedge product" is denoted by juxtaposition.) The separated orbits correspond to form (IV) and there is (essentially) only one invariant polynomial. The other forms may be defined through the use of covariants, where the definition of covariant is now extended to mean the invariants of SL(V) acting on $\bigwedge^k(V) \times V \times \ldots \times V$, where V appears up to 6 times. These facts - covariants define all canonical forms on $\bigwedge^k(V)$ but invariants do not - may be shown to hold quite generally. (The extended definition of covariant given above has its roots in a classical theorem of nineteenth century invariant theory known as Gram's theorem.)

A. Nijenhuis[1] ("The equivalence problem for tensor fields") considers a similar classification problem which arises in the study of differentiable manifolds. The objects to be classified are tensor fields (rather than skew -

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1 Added in print: This paper is not included in this volume as it is due to appear elsewhere.

symmetric tensors) and equivalence is defined with respect to diffeomorphisms (rather than SL(V)). Differential invariants play the role of invariant polynomials. Some examples of differential invariants are known but, as yet, there is no systematic procedure available for constructing them.

Question (C) : constructing invariants

Ideally, we would like (in some way) to explicitly describe some or all invariant polynomials. This is possible, for example, in Example 1 above. However, even in the case of binary forms, there are major difficulties. Mathematicians in the nineteenth century devised a symbolic technique for generating such invariants. Cayley's symbolism has its roots in the use of differential operators to produce invariants. The German symbolic technique, as developed by Aronhold, Clebsch, and Gordan, is algebraic and begins with the "symbolic" representation of an element f in V_d as $(ax + by)^d$. Eventually, it was shown that the basic invariants for binary forms are determinants symbolically. This has been extended now to arbitrary symmetric and skew-symmetric tensors.

P. Olver ("Invariant theory and differential equations") shows that several questions in differential equations can be attacked by a transform technique which changes differential operations into algebraic ones. This transform approach is closely related to the symbolic method. Furthermore, the notion of transvectant (as arising in the study of binary forms) can be generalized and becomes important in questions involving the divergence.

We noted earlier that $K[V]^G$ is always finitely generated when G is semi-simple; we now ask whether it is possible to explicitly display the generators of $K[V]^G$. In the case of binary forms, Hilbert attacked this problem in a totally unexpected way. Let $N_d = \{v \in V_d : f(v) = 0$ for every homogeneous, non-constant, polynomial $f \in \mathbb{C}[V_d]^G\}$. A form in N_d is called a <u>non-stable form</u>. Hilbert showed that there are finitely many invariant polynomials $f_1, \ldots, f_m$

so that $N_d = \{v \in V_d : f_i(v) = 0$ for $i = 1, \ldots, m\}$. Furthermore, for such a set of invariant polynomials, $\mathbf{C}[V_d]^G$ is integral over $\mathbf{C}[f_1, \ldots, f_m]$. Hence, the construction of generators for $\mathbf{C}[V_d]^G$ is closely related to the geometric study of N_d .

These concepts may be extended to the actions of arbitrary semi-simple groups on finite-dimensional vector spaces. G. Kempf ("Constructing invariants") explains the basic ideas and shows how they lead to explicit (though, huge) bounds on the degrees of the generators for $K[V]^G$.

The invariants in $K[V]^G$ may be constructed, once an explicit expression for a basis of V is known, through the technique of protomorphs ("Constructing invariants via Tschirnhaus transformations"). This technique comes from the classical theory of binary forms and gives information on $K[V]^G$ and, also, algebras of invariant polynomials for certain subgroups of G.

F. G.

INVARIANTS OF UNIPOTENT GROUPS

A survey

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I'll give a survey on the known results on finite generation of invariants for nonreductive groups, and some conjectures.

You know that Hilbert's 14th problem is solved for the invariants of reductive groups; see [12] for a survey. So the general case reduces to the case of unipotent groups. But in this case there are only a few results, some negative and some positive. I assume that k is an infinite field, say the complex numbers, but in most instances an arbitrary ring would do it.

1. BASIC RESULTS

a) <u>Nagata's counterexample</u> (1958): Let U be a subgroup of the n-fold product $G_a{}^n$ of the additive group, canonically embedded in GL_{2n}

$$U \leq \begin{pmatrix} \begin{smallmatrix}1&*\\0&1\end{smallmatrix} & & \\ & \ddots & \\ & & \begin{smallmatrix}1&*\\0&1\end{smallmatrix}\end{pmatrix} \leq GL_{2n} ,$$

such that U is given by 3 'general' linear relations. Then $k[X]^U$ is not finitely generated, where $X = (X_1, \ldots X_{2n})$, if n is a square $= r^2 \geq 16$ (at least if k contains enough transcendental elements), cf. [14]. All known counterexamples derive from this one!

Chudnovsky claims, but apparently never published a proof, that $n \geq 10$ suffices. The argument in [1] is not convincing, but there is more evidence in [2] and [15]. For the proof (with $n \geq 10$) one needs the following result:

- ▶ There is a set S of n points in the affine plane $\mathbb{A}^2$ with the property: Each nonzero polynomial $f \in k[Y_1, Y_2]$ that vanishes of order at least t in each $p \in S$ has degree $d > t \cdot \sqrt{n}$ (t any integer ≥ 1).

Let $\omega_t(S)$ be the minimum of the degrees of such polynomials; then the assertion is $\omega_*(S) > t\cdot\sqrt{n}$. Now the quotient $\omega_*(S)/t$ decreases to a limit $\Omega(S)$ when t goes to infinity; $\Omega(S)$ is called the singular degree of S. In general $\Omega(S) \leq n$. Chudnovsky's claim is:

▶ If S is generic, then $\Omega(S) = n$.

This gives $\omega_*(S) \geq t\cdot\sqrt{n}$. However, if n is not a square, we have the desired strict inequality because $\omega_*(S)$ is an integer. And in the case where n is a square, we can take Nagata's argument.

b) <u>Popov's theorem</u> (1979) is the converse of the invariant theorem for reductive groups. So for an affine algebraic group G the following statements are equivalent:

(i) G is reductive.
(ii) Whenever G acts rationally on a finitely generated algebra A, then the invariant algebra A^G is finitely generated.

See [19]. This means that for a nonreductive group there can't be a general positive answer.

c) A <u>positive result</u> goes back to <u>Zariski</u> (1954): If a group G acts on a finitely generated algebra A such that the invariant algebra A^G has transcendence degree at most 2, then A^G is finitely generated, cf. [14]. A useful geometric version is:

COROLLARY 1. If an affine algebraic group G acts on an affine variety X and there is an orbit of codimension ≤ 2, then $k[X]^G$ is finitely generated.

<u>Proof</u>. Assume (without loss of generality) that X is normal. Then

$$\operatorname{trdeg} k[X]^G \leq \dim X - \max\{\dim G\cdot x \mid x \in X\} \leq 2.$$

COROLLARY 2. If trdeg $A \leq 3$, then A^G is finitely generated.

<u>Proof</u>. Assume that G acts effectively. If G is finite, we are done. Else trdeg $A^G \leq 2$.

For linear actions we can do one more step:

COROLLARY 3. If G acts linearly on the polynomial algebra $k[X] = k[X_1,X_2,X_3,X_4]$, then $k[X]^G$ is finitely generated.

<u>Proof</u>. Assume that G acts effectively. Without changing $k[X]^G$ we may assume that G is Zariski-closed in GL_4. If G is finite, we are done. Else $\dim G \geq 1$ and G is reductive or has a 1-dimensional unipotent normal subgroup N. The algebra $A = k[X]^N$ is finitely generated by Weitzenböck's theorem (see below 2.1b), trdeg $A \leq 3$, $k[X]^G = A^G$.

d) Some other positive results derive from <u>Grosshans's principle</u> [6]: Let an algebraic group G act rationally on a k-algebra A, and H be a closed subgroup of G. Then

$$A^H \cong (k[G]^H \otimes A)^G.$$

For the <u>proof</u> let $G \times H$ act on $k[G] \otimes A$ as follows: G acts diagonally by left translation and H acts on k[G] by right translation. Then take the invariants in the two possible different ways (using an obvious isomorphism).

If G is reductive and A finitely generated, this reduces the question, whether A^H is finitely generated, to the <u>one</u> algebra $k[G]^H$ that is also the global coordinate algebra of the homogeneous space G/H.

2. APPLICATIONS OF THE GROSSHANS PRINCIPLE

For ring theoretic properties of A^H it may be useful to look at the isomorphism of 1.d. For example an unpublished result of Boutot is:

- Let char k = 0 and G reductive, acting on a finitely generated k-algebra B with only rational singularities. Then B^G also only has rational singularities; in particular B^G is Cohen-Macaulay.

The question whether $k[G]^H$ has rational singularities, seems to be rather difficult, and I don't dare making a conjecture; but there are some known examples. If that holds, and A only has rational singularities, then also $k[G]^H \otimes A$ and hence A^H only have rational singularities.

The Grosshans principle has several important special cases that were known earlier, but derived with more pains:

1.) Let $G = SL_2$ and H be the maximal unipotent subgroup consisting of upper triangular matrices. Then $k[G]^H$ is the coordinate algebra k[V] of the affine plane $V = \mathbb{A}^2$, because H is the stabilizer G_x of the point x = (1,0) whose orbit $G \cdot x = \mathbb{A}^2 - \{0\}$ is dense and isomorphic to G/H and has a boundary of codimension 2. Here are two interesting applications of this situation:

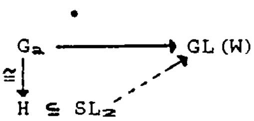
a) Let A be the coordinate algebra $k[R_d]$ of the vector space R_d of binary forms of degree d. Then we get the isomorphism

$$k[V \oplus R_d]^G \xrightarrow{\sim} k[R_d]^H$$

between 'covariants' and 'seminvariants', given by evaluating a covariant F at the point x,

$$F \longrightarrow F((1,0), -),$$

where the image is the 'Leitglied' (leading term) of the covariant. This result goes back to Roberts (around 1870).



b) Let char k = 0 and A be the coordinate algebra $k[W]$ of an arbitrary rational (finite dimensional) G_a-module W. Then the representation of G_a extends to SL_2 via the embedding by the Jordan normal form. Therefore the invariant algebra $k[W]^H \cong k[V \oplus W]^G$ is finitely generated. This is Seshadri's proof [20] of <u>Weitzenböcks theorem</u> (1932). Fauntleroy recently found a proof of this theorem in positive characteristic, see these conference proceedings or [5]. The proof is a skillful elaboration of the given one in characteristic 0 but, strictly speeking, doesn't depend on Grosshans's principle.

2.) Somewhat more generally we can take G reductive and H, a maximal unipotent subgroup of G. The principle for this case was observed by several people, for the first time (in characteristic 0) by Hadžiev [10], see also [6] and [21].

3.) Now let $G = GL_n$ act on the polynomial ring $k[X] = k[X_{ij} \mid 1 \le i,j \le n]$ in a matrix of indeterminates by left translation. Let H be a subgroup of SL_n such that $k[X]^H$ is finitely generated. Then for any affine algebra A on which GL_n (or a reductive group G between H and GL_n) acts rationally, the invariant algebra A^H is finitely generated. This is a qualitative version of the old principle: 'If you know the invariants of n vectors, you know all invariants.' (Capelli 1887) - The n vectors are the columns of the n-by-n matrix X.

The <u>proof</u> is two lines:

$$A^H \cong (k[GL_n]^H \otimes A)^{GL_n},$$

(I interchanged left and right translation, but that doesn't matter) and

$$k[GL_n]^H = k[X][1/\det]^H = k[X]^H[1/\det]$$

because $H \le SL_n$ and det is SL_n-invariant.

Note that we need an action of a bigger reductive group G containing H - of course this is a disadvantage, but in view of Nagata's counterexample it even looks surprisingly good.

3. GROSSHANS SUBGROUPS

The following seems to be a <u>good substitute of Hilbert's 14th problem</u>:

> Find the Grosshans subgroups of GL_n
> or more generally of a reductive group G.

The formal <u>definition</u> of a <u>Grosshans subgroup</u> H of an affine algebraic group is: H is closed, G/H is quasiaffine, k[G/H] is finitely generated. The technical condition 'G/H quasiaffine' is automatic if H is unipotent. Let me give 3 examples:

a) By Hadžievs result the maximal unipotent subgroups are Grosshans, even if G is not reductive.

b) The existence of non-Grosshans subgroups follows from Nagata's counterexample: There must be a situation

$$GLn \geq U \geq V, \quad U \text{ and } V \text{ unipotent with } \dim U/V = 1,$$

such that U is Grosshans and V is not.

c) Generic stabilizers often are Grosshans subgroups. The following theorem generalizes a result by Grosshans [7]. Since some people recently were interested in it, I give the proof here. I would be happy to see a substantial application.

THEOREM. <u>Let</u> X <u>be a factorial affine variety and</u> G, <u>an affine algebraic group acting on</u> X. <u>Then</u> X <u>has a dense open subset</u> U <u>such that the stabilizer</u> G_x <u>is a Grosshans subgroup of</u> G <u>for all</u> $x \in U$.

<u>Remark</u>. Instead of 'factorial' the following condition suffices: X is normal and each G-invariant divisor on X has finite order in the divisor class group Cl(X), cf. [18].

<u>Proof</u>. I may assume G connected. There is a function $f \in k[X]$ such that the principal open subset X_f is G-stable and $k(X)^G$ is the quotient field of $k[X_f]^G$; this is well-known, cf. [13]. Choose functions $f_1, \ldots, f_n \in k[X_f]^G$ that generate the field $k(X)^G$. Let R be the algebra generated by $f_1, \ldots, f_n$ and Y be an affine model of R. Then $k(Y) = k(X)^G$, and the induced morphism $\pi: X_f \longrightarrow Y$ is dominant.

Now let $m = \max\{\dim G\cdot x \mid x \in X\}$ be the maximal orbit dimension. The set $Z = \{x \in X_f \mid \dim G\cdot x = m\}$ is G-stable and open dense in X, and $\dim Y = \dim X - m$. Since X_f is factorial, $X_f - Z = V(h) \cup A$ for a function $h \in k[X_f]$ with $\dim A \leq \dim X - 2$. Clearly X_{fh} is G-stable. Restricting π gives a dominant morphism $\sigma: X_{fh} \longrightarrow Y$. The fibers of σ are G-stable, and

$$\sigma^{-1}\sigma x = (\sigma^{-1}\sigma x \cap Z) \cup (\sigma^{-1}\sigma x \cap A) \quad \text{for all } x \in Z.$$

There is a dense open part $W \subseteq Z$ such that $\sigma^{-1}y$ has pure dimension m for all $y \in W$. Shrinking W we may assume that

$$\dim(\sigma^{-1}y \cap A) \leq m-2 \quad \text{for all } y \in W.$$

Now $U = Z \cap \sigma^{-1}W$ is G-stable and open dense in X. Let $x \in U$. Then the closure $\overline{G\cdot x}$ in X_{rn} is an irreducible component of $\sigma^{-1}\sigma x$ - compare the dimensions. In $\overline{G\cdot x} - G\cdot x$ there is no $z \in Z$ since there is no room for the m-dimensional orbit $G\cdot z$. Thus $\overline{G\cdot x} - G\cdot x \subseteq \sigma^{-1}\sigma x \cap A$, and

$$\dim(\overline{G\cdot x} - G\cdot x) \leq \dim(\sigma^{-1}\sigma x \cap A) \leq m - 2 \leq \dim G\cdot x - 2.$$

Therefore G_x is a Grosshans subgroup of G. ∎

There are some natural conjectures:

Conjecture m: Each m-dimensional unipotent subgroup is Grosshans.

This conjecture is false when $m = r^2 - 3$ and $r \geq 4$, and probably false when $m \geq 7$. Conjecture 1 is true by Weitzenböck's theorem, and I guess this is the only positive case! Since nobody seems to have an approach to this problem, I make another conjecture:

Conjecture A: Each regular unipotent subgroup of a reductive group is Grosshans.

'Regular' means 'normalized by a maximal torus', or, more concretely, given by a closed subset of the root system. For GL_n it means that the subgroup is defined by relations of the type $X_{ij} = 0$. The following example shows what this means:

$$\begin{bmatrix} 1 & 0 & * & 0 \\ & 1 & 0 & * \\ & & 1 & 0 \\ & & & 1 \end{bmatrix} = \left\{ \begin{bmatrix} 1 & 0 & a & 0 \\ & 1 & 0 & b \\ & & 1 & 0 \\ & & & 1 \end{bmatrix} \mid a, b \in k \text{ arbitrary} \right\}.$$

Such a pattern of zeroes and stars above the diagonal gives a subgroup of GL_n, if and only if it is the incidence matrix of a strict ordering of the set $\{1,...,n\}$.

Conjecture A is true for the unipotent radicals $H = R_u(P)$ of the parabolic subgroups P. This was shown by Hochschild and Mostow 1973 (for characteristic 0) [11], and by Grosshans 1983 (for the general case) [8]. Grosshans recently extended this result in several ways [9].

4. INVARIANT MINORS

My own contribution in [16], [17] is a large class of examples for GL_n - but unfortunately I have no general proof of conjecture A, not even for GL_n. My approach is the explicit determination of $k[X]^H$, where $X = (X_{ij})$ is the n-by-n matrix of indeterminates and $H \leq GL_n$ regular unipotent. It looks promising because a lot of invariants are obvious: Consider a minor

$$\begin{vmatrix} X_{i_1 j_1} & \cdots & X_{i_1 j_m} \\ \vdots & & \vdots \\ X_{i_m j_1} & \cdots & X_{i_m j_m} \end{vmatrix},$$

shortly represented by the row $(i_1 \ldots i_m | j_1 \ldots j_m)$. When is it invariant? For the group at the end of section 3 we have

$$\begin{bmatrix} 1 & 0 & a & 0 \\ & 1 & 0 & b \\ & & 1 & 0 \\ & & & 1 \end{bmatrix} \cdot \begin{bmatrix} X_{11} & \ldots \\ X_{21} & \ldots \\ X_{31} & \ldots \\ X_{41} & \ldots \end{bmatrix} = \begin{bmatrix} X_{11}+aX_{31} & \ldots \\ X_{21}+bX_{41} & \ldots \\ X_{31} & \ldots \\ X_{41} & \ldots \end{bmatrix}.$$

So our minor is invariant if and only if the following is true:

- If it contains the row index 1, then it also contains 3, if it contains the row index 2, then it also contains 4.

This is because then H acts by elementary row operations. In general we read the condition off the constellation of stars:

- If the box representing the group contains stars at positions $l_1, \ldots, l_m$ in the i-th row, then the minor has to contain the row indices $l_1, \ldots, l_m$ along with i.

So we know the invariant minors. I would like to prove:

<u>Conjecture B</u>: The invariant algebra $k[X]^H$ is generated by the (finitely many) invariant minors.

This would imply Conjecture A for GL_n. In fact I can prove a much stronger result, but only for a large class of unipotent subgroups that however contains the unipotent radicals of the parabolics as simplest special cases. Since the proof (and even the statement of the result) uses some complicated combinatorial methods, I'll give only a very simple example that, of course, is not new.

Take n = 2 and H, the maximal unipotent subgroup consisting of upper triangular matrices. The invariant minors are:

$$(1\ 2|1\ 2) = \det, \quad (2|1) = X_{21}, \quad (2|2) = X_{22},$$

because, if we have the row index 1 we also must have the row index 2, so the minor is the full determinant. Now $k[X][1/X_{22}] = R[1/X_{22}][X_{12}]$, where R is the algebra generated by the invariant minors; H acts trivially on $R[1/X_{22}]$ and maps X_{12} to $X_{12} + sX_{22}$ with $s \in k$ arbitrary. Therefore

$$k[X][1/X_{22}]^H = R[1/X22].$$

This kind of argument goes through for general n: There is a product ϵ of invariant minors such that $k[X][1/\epsilon]^H = R[1/\epsilon]$. This means that the analogue of Conjecture B for rational functions is true. However, as it is often the case, it is a major problem to get rid of this denominator.

Let me continue the example. We have

$$k[X]^H = k[X] \cap R[1/X_{22}] \supseteq R.$$

To get equality I have to show: If $f \in k[X]$ and $X_{22}f \in R$, then $f \in R$ - by induction I may assume $r = 1$. This is the hard core of the proof, that however is no problem for this example. Write a product of minors in the form of a bitableau, say

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 2 \\ 1 & & 1 & \\ 1 & & 2 & \\ 2 & & 1 & \end{array}\right].$$

These bitableaux span $k[X]$. When the columns increase, we have a 'standard' bitableau. The given one is not standard because of its last entry. The straightening law by Rota and others, see [3] for example, says (for the general case of n-by-n matrices):

► The standard bitableaux are a basis of $k[X]$.

This generalizes some classical determinant identities. In the example we have

$$\left(\begin{array}{c|c} 1 & 2 \\ 2 & 1 \end{array}\right) = \left(\begin{array}{c|c} 1 & 1 \\ 2 & 2 \end{array}\right) - (1\ 2 | 1\ 2);$$

in a similar way each bitableau obviously is a linear combination of standard ones.

Now call a bitableau 'admissible', if each of its rows represents an invariant minor. Then R is spanned by the admissible bitableaux. In our example the admissible standard bitableaux are a basis of R: 'Admissible' means that there is no row $(1|\ldots)$; but then the bitableau is already standard.

Now take $f \in k[X]$ such that $X_{22}f \in R$, and write it as a linear combination of standard bitableaux: